GROUND WATER MOVEMENTS AT WATER LEVEL FLUCTUATIONS IN A RESERVOIR WITH A VERTICAL BOUNDARY

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PMM Vol.23, No.3, 1959, pp. 540-545

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(Received 12 February 1959)

The problem of ground water movement in soil of infinite depth due to the sinusoidal fluctuations of water level in a reservoir was treated by Meyer [1] and also by Carrier and Munk [2]. Both investigations presumed that the free surface of ground flow varies slowly and the condition on it becomes linear. Another method of investigation of such problems is given in this article, using the Laplace transform, which allows some generalization.

1. A movement in a vertical plane xz is investigated here. As is known from the theory of ground water movement the velocity of filtration has a potential $\phi(x, z, t) = -kh(x, z, t)$, where k is a constant for homogeneous soil (filtration coefficient), $h(x, z, t)$ is the pressure function or pressure head. The function $h(x, z, t)$ satisfies the Laplace equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} = 0 \tag{1.1}$$

The axis z is assumed directed vertically down. The following conditions must be fulfilled on the free surface [1,3]

$$z + h(x, z, t) = 0 \tag{1.2}$$

$$\frac{\partial h}{\partial t} - \frac{k}{m}\frac{\partial h}{\partial z} + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2 = 0 \quad (m \text{ is the porosity}) \tag{1.3}$$

In a linearized treatment the condition (1.3) is replaced by

$$\frac{\partial h}{\partial t} - c\frac{\partial h}{\partial z} = 0 \qquad \left(c = \frac{k}{m}\right) \tag{1.4}$$

and transferred to the plane $z = 0$. Instead of the condition (1.2) the following equation is obtained for the free surface:

$$z = -h(x, 0, t) \tag{1.5}$$

The Laplace transform is introduced

$$Lh(x, z, t) = H(x, z, p) = \int_0^\infty e^{-pt} h(x, z, t)\, dt \tag{1.6}$$

Then, instead of (1.4) we have

$$\frac{\partial H}{\partial z} - \frac{p}{c} H = -\frac{1}{c} h(x, 0, 0) \qquad \text{for } z = 0 \tag{1.7}$$

where $z = -h(x, 0, 0)$ is the initial form of a free surface.

We investigate the function

$$\Phi(x, z, p) = \frac{\partial H}{\partial z} - \frac{p}{c} H \tag{1.8}$$

It satisfies the Laplace equation for the variables x, y, z in the field of motion. When $z = 0$ the following condition must be fulfilled:

$$\Phi(x, 0, p) = -\frac{1}{c} h(x, 0, 0) \tag{1.9}$$

The function $\Phi(x, z; p)$ has the following properties. If H = const along any part of a vertical line, then $\partial H/\partial z = 0$ and Φ = const. Similarly, when a part of the vertical line is a solid wall, along it $\partial H/\partial x = 0$ and therefore along the wall

$$\frac{\partial \Phi}{\partial x} = \frac{\partial}{\partial z}\left(\frac{\partial H}{\partial x}\right) - \frac{p}{c}\frac{\partial H}{\partial x} = 0$$

A case is discussed first, when the free surface of ground water is horizontal at the beginning, namely $h(x, y, 0, 0) = 0$. Then the function $\Phi(x, z; p)$ may be assumed as a potential of a fictitious velocity in the field limited by a horizontal plane ($z = 0$) of constant potential, and also by vertical planes of constant potential and vertical solid walls.

Assume that we found the function $\Phi(x, z; p)$. After integration of (1.8) we shall have

$$H(x, z; p) = \exp\frac{pz}{c}\int_{z_1}^{z} \exp\frac{-p\zeta}{c}\,\Phi(x, \zeta; p)\, d\zeta + A(x; p)\exp\frac{pz}{c} \tag{1.10}$$

The function $A(x; p)$ is to be selected by the condition that $H(x, z; p)$ satisfies the Laplace equation. This leads to the equation

$$\frac{d^2 A}{dx^2} + \frac{p^2}{c^2} A = -\left[\frac{\partial \Phi}{\partial z}\right]_{z=0} + \frac{p}{c}\Phi(x, 0; p) \tag{1.11}$$

2. The following problem is discussed as an example. The boundary between reservoir and soil is a vertical plane $x = 0$. The water level varies in time according to a given law, so that a pressure function $h(x, z, t)$ is given for $x = 0$:

$$h(0, z, t) = f(t) \tag{2.1}$$

Let us denote by $F(p)$ the Laplace transform $Lf(t)$:

$$Lf(t) = F(p) = \int_0^\infty e^{-pt} f(t)\, dt \tag{2.2}$$

Then we may put

$$H(0, z;\ p) = F(p) \tag{2.3}$$

Obviously, $\partial H/\partial x = 0$ when $x = 0$, therefore for $x = 0$

$$\Phi(0, z;\ p) = -\frac{p}{c} F(p) \tag{2.4}$$

The initial form of the free surface is assumed horizontal, therefore, we have for $z = 0$

$$\Phi(x, 0;\ p) = 0 \tag{2.5}$$

It is not difficult to find the harmonic function $\Phi(x, z;\ p)$ in the quadrant $x > 0$, $z > 0$, which has constant values 0 and $-p/c\ F(p)$ at the sides of this quadrant. Namely

$$\Phi(x, z;\ p) = -\frac{2p}{\pi c} F(p) \operatorname{arc\,tg} \frac{z}{x} \tag{2.6}$$

Substituting the obtained expression $\Phi(x, z;\ p)$ into equation (1.11)

$$\frac{d^2 A}{dx^2} + \frac{p^2}{c^2} A = \frac{2p\,F(p)}{\pi c} \cdot \frac{x}{x^2 + z^2}\bigg|_{z=0} = \frac{2pF(p)}{\pi c x} \tag{2.7}$$

we then obtain a general solution of this equation:

$$A(x;\ p) = +\frac{2}{\pi} F(p) \int_x^\infty \frac{\sin p(\xi - x)/c}{\xi} d\xi + C_1 \sin\frac{px}{c} + C_2 \cos\frac{px}{c} \tag{2.8}$$

Note that the integral in (2.8) can be expressed in the form

$$\int_x^\infty \frac{\sin[p(\xi - x)/c]}{\xi} d\xi = \int_0^\infty \frac{\sin(p\zeta/c)}{\zeta + x} d\zeta \tag{2.9}$$

Now the expression for $H(x, z;\ p)$ can be derived from the formula (1.10) taking into consideration (2.8):

$$H(x, z;\ p) = \tag{2.10}$$

$$= -\frac{2pF(p)}{\pi c} \exp\frac{pz}{c} \int_0^z \exp\frac{-p\zeta}{c} \operatorname{arc\,tg}\frac{\zeta}{x}\, d\zeta + \frac{2}{\pi} F(p) \exp\frac{pz}{c} \int_0^\infty \frac{\sin(p\xi/c)}{\xi + x} d\xi$$

The arbitrary constants z_1 in the formula (1.10), c_1 and c_2 in the formula (2.8) are assumed equal zero since for $z = 0$ we have $H(0, z; p) = F(p)$, and for $z = 0$, $x = \infty$ we have $H(\infty, 0; p) = 0$.

The first integral in (2.10) is replaced by the sum of two integrals with limits $(0, \infty)$ and (∞, z). The integral from 0 to ∞ can be represented as:

$$\int_0^\infty \exp\frac{-p\zeta}{c}\operatorname{arc\,tg}\frac{\zeta}{x}\,d\zeta = -\frac{c}{p}\left(ci\frac{px}{c}\sin\frac{px}{c} + si\frac{px}{c}\cos\frac{px}{c}\right) \equiv \frac{c}{p}M(x, p) \tag{2.11}$$

where

$$cix = \int_x^\infty \frac{\cos t}{t}\,dt, \qquad six = -\int_x^\infty \frac{\sin t}{t}\,dt \tag{2.12}$$

On the other hand, we have [4]

$$\int_0^\infty \frac{\sin(p\xi/c)}{\xi + x}\,d\xi = -ci\frac{px}{c}\sin\frac{px}{c} - si\frac{px}{c}\cos\frac{px}{c} = M(x; p) \tag{2.13}$$

Introducing the relations (2.11) and (2.13) into (2.10) we have

$$H(x, z; p) = \frac{2pF(p)}{\pi c}\int_z^\infty \exp\frac{-p(\zeta - z)}{c}\operatorname{arc\,tg}\frac{\zeta}{x}\,d\zeta \tag{2.14}$$

Integration by parts gives

$$H(x, z; p) = \frac{2}{\pi}F(p)\operatorname{arc\,tg}\frac{z}{x} - \frac{2x}{\pi}F(p)\int_\infty^z \frac{d\zeta}{\zeta^2 + z^2}\exp\frac{-p(\zeta - z)}{c\zeta} \tag{2.15}$$

As known from operational calculus, when $F(p) = Lf(t)$, then

$$e^{-pa}F(p) = Lf_1(t), \qquad f_1(t) = \begin{cases} f(t-a) & (t > a) \\ 0 & (t < a) \end{cases} \tag{2.16}$$

In the discussed case

$$f_1(t) = \{f[t - (\zeta - z)/c]\} \qquad \text{for } t > (\zeta - z)/c, \text{ т. е. } \zeta < z + ct$$
$$f_1(t) = 0 \qquad \text{for } \zeta > z + ct$$

So the following relation is obtained for the unknown function $h(x, z, t)$

$$h(x, z; t) = \frac{2}{\pi}f(t)\operatorname{arc\,tg}\frac{z}{x} + \frac{2x}{\pi}\int_z^{z+ct} f\left(t - \frac{\zeta - z}{c}\right)\frac{d\zeta}{\zeta^2 + x^2} =$$
$$= \frac{2}{\pi}f(t)\operatorname{arc\,tg}\frac{z}{x} + \frac{2xc}{\pi}\int_0^t \frac{f(t-u)\,du}{(z + ct)^2 + y^2} \tag{2.17}$$

It is easy to check that $h(0, z, t) = f(t)$ for $x = 0$, and $h = 0$ for $z = 0$ and $t = 0$.

The equation of the free surface is according to (1.5):

$$z = -h(x, 0, t) = -\frac{2x}{\pi}\int_0^{ct} f\left(t - \frac{\zeta}{c}\right)\frac{d\zeta}{\zeta^2 + x^2} = -\frac{2xc}{\pi}\int_0^t \frac{f(t-u)\,du}{x^2 + c^2u^2} \tag{2.18}$$

First example. Let $f(t) = at$, $0 < t < T$.

The pressure function is from formula (2.171

$$h(x, z, t) = \frac{2}{\pi} at \operatorname{arc\,tg} \frac{z}{x} + \\ + \frac{2a}{\pi c}(z + ct)\left[\operatorname{arc\,tg}\frac{z + ct}{x} - \operatorname{arc\,tg}\frac{z}{x}\right] - \frac{xa}{\pi c}\ln\frac{(z + ct)^2 + x^2}{z^2 + x^2} \tag{2.19}$$

The equation of the free surface is

$$z = -h(x, 0, t) = -\frac{2at}{\pi}\operatorname{arc\,tg}\frac{ct}{x} + \frac{xa}{\pi c}\ln\frac{x^2 + c^2t^2}{x^2} \tag{2.20}$$

Second example. Let $f(t) = -H_0 \cos\omega t$. Then

$$f\left(t - \frac{\zeta - z}{c}\right) = -H_0\cos\omega\left(t + \frac{z}{c} - \frac{\zeta}{c}\right) = \\ = -H_0\cos\omega\left(t + \frac{z}{c}\right)\cos\frac{\zeta}{c} - H_0\sin\omega\left(t + \frac{z}{c}\right)\sin\frac{\zeta}{c}$$

From formula (2.17) we get

$$h(x, z, t) = -\frac{2H_0}{\pi}\cos\omega t \operatorname{arc\,tg}\frac{z}{x} - \frac{2H_0x}{\pi}\cos\omega\left(t + \frac{z}{c}\right)\int_z^{z+ct}\frac{\cos(\omega\zeta / c)\,d\zeta}{\zeta^2 + x^2} - \\ - \frac{2H_0x}{\pi}\sin\omega\left(t + \frac{z}{c}\right)\int_z^{z+ct}\sin\frac{\omega\zeta}{c}\frac{d\zeta}{\zeta^2 + x^2} \tag{2.21}$$

The equation of the free surface is

$$z = -h(x, 0, t) = \frac{2H_0x}{\pi c}\left\{\cos\omega t\int_0^t\frac{\cos\omega u\,du}{u^2 + x^2/c^2} + \sin\omega t\int_0^t\frac{\sin\omega u\,du}{u^2 + x^2/c^2}\right\} \tag{2.22}$$

For large values of t this expression can be transformed, dividing the limits of integration $(0, t)$ into two parts $(0, \infty)$ and (∞, t) and taking in account relations

$$\int_0^\infty\frac{\cos\omega u\,du}{u^2 + x^2/c^2} = \frac{\pi c}{2x}\exp\left(\frac{-\omega x}{c}\right) \tag{2.23}$$

$$\int_0^\infty \frac{\sin \omega u du}{u^2 + x^2 / c^2} = \frac{c}{2x}\left[\exp\frac{-\omega x}{c}\,\overline{\mathrm{Ei}}\left(\frac{\omega x}{c}\right) - \exp\frac{\omega x}{c}\,\mathrm{Ei}\left(-\frac{\omega x}{c}\right)\right] \equiv \frac{c}{2x} N(x) \qquad (2.24)$$

Here

$$\overline{\mathrm{Ei}}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t}\, dt \quad (x > 0), \qquad \mathrm{Ei}(-x) = -\int_x^\infty \frac{e^{-t}dt}{t} \quad (x > 0)$$

(the first integral is taken as a Cauchy principal value). Now, instead of (2.22) we have

$$z = H_0 \left\{\exp\frac{-\omega z}{c}\cos\omega t + \frac{1}{\pi} N(x) \sin\omega t - \frac{2x}{\pi c}\int_t^\infty \frac{\cos\omega u du}{u^2 + x^2 / c^2} - \frac{2x}{\pi c}\int_t^\infty \frac{\sin\omega u du}{u^2 + x^2 / c^2}\right\} \qquad (2.25)$$

The last two terms are small for sufficiently large values of t and the sum of the two first items gives "quasi-steady" oscillations of the ground water surface.

3. Our results can be compared with the Meyer's solution [1] for the problem of ground water motion due to the sinusoidal water fluctuations in a reservoir. Meyer seeks a solution of the form

$$h(x, z, t) = \mathrm{Re}\,[h^\circ(x, z)\, e^{i\omega t}] \qquad (3.1)$$

In the sequence the symbol Re of the real part is omitted.

The condition (1.4) for the free surface is to be rewritten (because Meyer's notation is different, as is also the direction of the axis z):

$$\frac{\partial h^\circ}{\partial z} - i\alpha h^\circ = 0 \qquad \left(\alpha = \frac{m\omega}{k}\right) \qquad (3.2)$$

A new function is introduced

$$\Phi^\circ(x, z) = \frac{\partial h^\circ}{\partial z} - i\alpha h^\circ \qquad (3.3)$$

which has the following properties: it equals zero when $z = 0$ and has a constant value $+ i\alpha H_0$ when $x = 0$, because it is assumed that

$$h(0, z, t) = -H_0 \cos\omega t \qquad (3.4)$$

and hence $h(0, z) = -H_0$.

Now we have

$$\Phi^\circ(x, z) = -\frac{2\alpha i}{\pi} H_0 \operatorname{arc\,tg} \frac{z}{x} \qquad (3.5)$$

After substituting this expression into (3.4) and carrying out the integration we have

$$h^{\circ}(x, z) = -\frac{2\alpha i}{\pi} H_0 e^{+i\alpha z} \int_1^z e^{-i\alpha u} \operatorname{arc\,tg} \frac{u}{x}\, du - \frac{2\alpha i}{\pi} H_0 e^{+i\alpha z} f(x) \qquad (3.6)$$

Transformation to dimensionless variables

$$\alpha z = \zeta, \qquad \alpha x = \xi, \qquad \alpha u = v$$

and consideration that $h^0(x, z)$ shall be a harmonic function in the quadrant $x > 0$, $z > 0$, leads to the equation

$$f'' - f = \frac{1}{\xi} \qquad (3.7)$$

Integration of this equation, taking account of the boundary conditions $f(\xi) = 1$ for $\xi = 0$ and $h^0(\infty, 0) = 0$, gives

$$f(\xi) = -\frac{1}{2} e^{\xi} \int_{\xi}^{\infty} \frac{e^{-w}}{w}\, dw + \frac{1}{2} e^{-\xi} \int_{-\xi}^{\infty} \frac{e^{-w}}{w}\, d\xi + \pi i e^{-\xi} = \frac{1}{2} N(\xi) + \pi i e^{-\xi}$$

(here the second integral assumes a Cauchy principal value).

Now we have for the $h^0(\xi, \zeta)$

$$h^{\circ}(\xi, \zeta) = -\frac{2iH_0}{\pi} e^{i\zeta} e^{-iv} \operatorname{arc\,tg} \frac{v}{\xi}\, dv - \frac{iH_0}{\pi} e^{i\zeta} N(\xi) - H_0 e^{i\zeta} e^{-\xi}$$

For $\zeta = 0$ it is

$$h^{\circ}(\xi, 0) = -H_0 \left[e^{-\xi} + \frac{i}{\pi} N(\xi) \right]$$

Because

$$h(\xi, 0, t) = \operatorname{Re}[h(\xi, 0) e^{i\omega t}] = -H_0 \left[e^{-\xi} \cos \omega t - \frac{1}{\pi} N(\xi) \sin \omega t \right]$$

the equation of the free surface is

$$z = -h(\xi, 0, t) = H_0 \left[e^{-\xi} \cos \omega t - \frac{1}{\pi} N(\xi) \sin \omega t \right]$$

Comparing this expression with that derived above (2.25), we can see that the Meyer's solution corresponds to the "quasi-steady" fluctuations of the ground water level. This can be explained by the fact that Meyer treated the problem without boundary conditions and for $t = 0$ in his solution the equation of the free surface $z = H_0 \exp(-\xi)$, is not an assumed *a priori*, but follows from the solution. In our problem, however, the condition $h(x, 0, 0) = 0$ was presumed.

4. Let us now assume, that the initial form of the free surface differs from zero, and $h(x, 0, 0)$ is a given function $\psi(x)$, so that the equation of the free surface for $t = 0$ is

$$z = -h(x, 0, 0) = \psi(x) \qquad (x > 0) \tag{4.1}$$

The field of our motion is the lower right quadrant of the plane xz. Let us extend the function $h(x, z, t)$ into the left quadrant, and assume that $h(x, 0, 0) = -h(-x, 0, 0)$ when $x < 0$. Then the integral

$$h_1(x, z, t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{h(\zeta, 0, 0)(z - c^2)\, d\zeta}{(z - ct)^2 + (x - \zeta)^2} =$$

$$= \frac{z - ct}{\pi}\int_0^{\infty} \frac{\psi(\zeta)\, d\zeta}{(z - ct)^2 + (x + \zeta)^2} - \frac{z - ct}{\pi}\int_0^{\infty} \frac{\psi(\zeta)\, d\zeta}{(z - ct)^2 + (x - \zeta)^2} \tag{4.2}$$

gives a harmonic function of x, z in the lower semiplane. This function becomes zero for $x = 0$ and becomes $-\psi(x)$ when $x = 0$ for $z = 0$, $t = 0$. Hence it follows that $h(x, z, t) + h_1(x, z, t)$, where h is determined by formula (2.17), h_1 by formula (4.2), gives the solution of the problem of ground water oscillations in the lower quadrant of the plane xz for a given law of the variation of the water level in a reservoir with time, and for a given initial form of the free surface.

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Translated by S.K.